Generalizing the classical fixed-centres problem in a non-Hamiltonian way

2004 J. Phys. A: Math. Gen. 37 9109

(http://iopscience.iop.org/0305-4470/37/39/005)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 9109–9123 PII: S0305-4470(04)78347-X

Generalizing the classical fixed-centres problem in a non-Hamiltonian way

A Albouy[1] **and T J Stuchi**[2]

[1] IMCCE, CNRS, 77, av. Denfert-Rochereau, 75014 Paris, France
[2] Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528 21945-970, Rio de Janeiro, Brazil

E-mail: albouy@imcce.fr and tstuchi@if.ufrj.br

Received 25 March 2004, in final form 26 July 2004
Published 15 September 2004
Online at stacks.iop.org/JPhysA/37/9109
doi:10.1088/0305-4470/37/39/005

Abstract

The problem of two gravitational (or Coulombian) fixed centres is a classical integrable problem, stated and integrated by Euler in 1760. The integrability is due to the unexpected first integral G. We introduce some straightforward generalizations of the problem that still have the generalization of G as a first integral, but do not possess the energy integral. We present some numerical integrations showing the main features of their dynamics. In the domain of bounded orbits the behaviour of these *a priori* non-Hamiltonian systems is very similar to the behaviour of usual near-integrable systems.

PACS number: 02.30.Ik

1. Introduction to affine dynamics

The word 'geometry' in its primitive understanding refers to the study of figures in the Euclidean plane or Euclidean space. Several 'non-Euclidean' extensions have been given. In particular, the consideration of planar transformations of the form $(x, y) \mapsto (\alpha x, \beta y)$, with $\alpha > 0$ and $\beta > 0$, named 'affinities' in Euler's *Introductio in analysin infinitorum*, grounds the 'affine geometry'. The publication of Möbius' *Der Barycentrische Calcul* (1827) and of Weyl's *Raum-Zeit-Materie* (1918) marked its development. It is not so easy to get a good intuition of affine space. The following explanation may be helpful.

Suppose we take on a blackboard standard coordinates x and y, i.e. coordinates related to an orthonormal frame, with a horizontal and a vertical axis. Suppose we define the distance between two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ by the formula

$$\mathrm{d}(A, B) = \sqrt{p(x_A - x_B)^2 + 2q(x_A - x_B)(y_A - y_B) + r(y_A - y_B)^2}, \qquad (1)$$

where p, q and r are fixed real numbers such that $pr > q^2$ and $p > 0$. It is well-known that, provided we find a mechanical or optical device to draw the strange 'circles' and to measure the new angles, we will be able to present to the students the usual constructions and propositions of Euclidean geometry. We will still draw the straight lines using the usual ruler.

This being observed, the usual choice $p = r = 1, q = 0$ appears as induced by the 'physical world'. From the point of view of the internal consistency of Euclidean geometry, no choice of (p, q, r), satisfying the inequalities above, is distinguished. In the mathematical terminology, choosing (p, q, r) we fix a Euclidean structure on the blackboard. Different values of (p, q, r) correspond to different Euclidean structures, which share an important property: they are such that the length is invariant by the usual translations.

To study the affine geometry of the blackboard means to decide not to make any choice among the formulae parametrized by p, q and r. In particular, there is no way to measure a distance or, which comes to the same thing, there is an infinity of non-equivalent ways to do so. We cannot compare the length of two segments, except those that have the same direction. The figures called circle, square, rhombus are not defined in affine geometry, while the ellipse, the hyperbola, the parabola, the parallelogram and the trapezoid are well-defined.

Affine geometry is somewhat simpler and more primitive than Euclidean geometry. In physics, it is still an abstract idea, because the spaces we need to visualize when we work with physical models are not purely affine. However, once we observe and teach that the spacetime of special relativity is an affine space (of course it has also a Minkowskian structure), we give quite an efficient description of it. Our intuition can rest on some of the concepts of the usual geometry.

Newton described geometry as a chapter of mechanics. While the figures are objects studied by geometry, the moving particles and the moving figures are objects studied in mechanics or dynamics. For example, the theory of geodesic lines should be a chapter of dynamics, not of geometry. Of course such a 'classification' does not survive to a mere change of our point of view. However, it suggests that we should study as well 'affine dynamics', 'projective dynamics', 'spherical dynamics', 'pseudo-Riemannian dynamics', etc.

Affine dynamics has not been studied, as far as we know. Historically, this is understandable, because at the time when affine geometry appeared, mechanics was too busy with the study of Hamilton equations, which are related at the beginning with Euclidean or pseudo-Riemannian dynamics, and subsequently quite incompatible with the refusal to choose a Euclidean structure. We will show in this paper that a classical system, the two-fixed-centres problem, may be generalized in two 'orthogonal' ways. The known one remains Euclidean, while the new one is affine and non-Euclidean (indeed something different replaces the Euclidean structure). The whole structure of the classical problem appears naturally as the superposition of the structures of both generalizations.

2. Historical introduction to the two-fixed-centres problem

In 1760, Euler stated and reduced to quadratures the two-fixed-centres problem. He considered a particle in the plane subject to the gravitational attraction of two fixed points with masses m_A and m_B. To write the corresponding differential system, we denote by q the position of the moving particle, by A and B the positions of the fixed attracting masses and by $q_A = q - A$ the vector joining A to q. We also make $q_B = q - B$ and $u = q_B - q_A = A - B$. The second derivative of q with respect to time is:

$$\ddot{q} = -m_A \|q_A\|^{-3} q_A - m_B \|q_B\|^{-3} q_B. \tag{2}$$

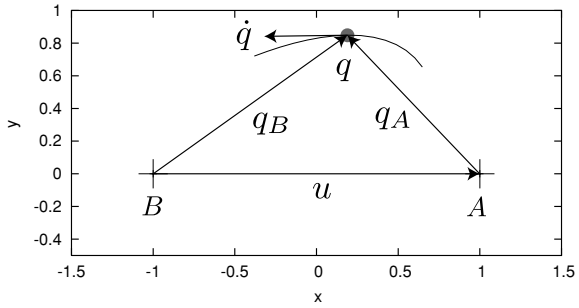

Figure 1. The two-fixed-centres problem. A particle q with coordinates (x, y) evolves under the attraction of two centres A and B, with respective coordinates $(1, 0)$ and $(-1, 0)$ and respective masses m_A and m_B.

The problem is rather artificial and Euler was interested in it because it is quite similar to what we now call the restricted three-body problem, which Euler was still hoping to integrate[3]. After finding two first integrals, Euler obtained new coordinates (r, s) for q which are separated in the new differential equation for the trajectories. The solutions could then be obtained by a quadrature, i.e. by the integration of some elementary function of one variable. In this case the function is a polynomial of degree 3 to the power $-1/2$, and the quadrature gives an elliptic integral. In 1763, Euler solved equation (2) in dimension 3. He also simplified his process of separating variables, introducing the so-called elliptic coordinates[4] $\sigma = \|q_A\| + \|q_B\|$, $\tau = \|q_A\| - \|q_B\|$. In 1767, Lagrange wrote, independently of this second work of Euler[5], a work where he also solved the 3D case using the elliptic coordinates. Furthermore, he noticed that Lambert's expression of the time in a Keplerian motion could be deduced restricting some formulae to the particular (and trivial) case $m_B = 0$. Finally, he showed that system (2) can still be integrated if we add the term $-m_O q_O$ to its second member, O being the middle point of (A, B), and $q_O = q - O$. In 1825, Legendre published 129 pages of detailed studies of the problem in his famous *Traité des fonctions elliptiques*. In particular, Euler and Legendre discussed the various cases where the particle describes an algebraic curve. Liouville discovered that if the second member of (2) is the gradient of a force function of the form $[\phi(\sigma) - \psi(\tau)]/(\sigma^2 - \tau^2)$, where ϕ and ψ are two functions, it is possible to integrate with the same method. System (2) with such a generalized second member still possesses two independent quadratic first integrals, one being the energy. Note that Darboux [Dar] established a kind of converse, starting with hypotheses on the first integrals, and arriving at Liouville's second member.

In 1885, Killing[6] extended the two fixed centres problem, changing the Euclidean space in any space of constant curvature. He obtained the separation of variables in sphero-elliptic coordinates[7] and similar coordinates if the curvature is negative. Developing the ideas of

[3] Euler wrote to Lagrange in 1768: 'Si vous avez réussi de donner à l'un des deux centres de force un mouvement autour de l'autre, quoiqu'il ne fût que circulaire et uniforme, je le regarderai comme la plus importante découverte dans l'Astronomie.' [Eu2]

[4] [Eu1], p 257; $\|u\|^{-1}\sigma = (1 - r)^{-1}(1 + r)$ and $\|u\|^{-1}\tau = (1 + s)^{-1}(1 - s)$; note that Legendre preferred (p, q) such that $s = p^2$ and $r = q^2$, while Liouville and Whittaker chose (α, β) with $\sigma = \|u\| \cosh\alpha$ and $\tau = \|u\| \cos\beta$.

[5] See [Lag], p 94. Euler had informed Lagrange in 1762 (see [Eu2]) that he generalized his integration to the 3D case.

[6] [Kil]. Thanks to C Velpry and A Borisov who indicated this paper to us. See also [KoH].

[7] The general case where this separation occurs was already published by Liouville [Lio], p 370. But Liouville did not point out the two-fixed-centres problem on the sphere.

[Ap1], one can show that Killing's problem corresponds to Euler's problem by a central projection (i.e. a projection from the centre of the sphere to the plane), combined with some affine transformation.

For a selection of important works on the two-fixed-centres problem and results about its integration see [Dem]. The problem has been considered as a useful starting point in astronautics and chemistry (for selected references see [HoW]). Making imaginary the distance between the two centres A and B, one approximates the gravitational attraction of an oblate earth (see [Bel]). Some regularities in the spectrum of the ion H_2^+ and of other ions or atoms may be understood starting from a quantization of Euler's problem (see [RoB]).

3. Generalizing the first integrals

We are only interested in trying to understand the integrability of (2) and we will restrict ourselves to the planar case. The main fact is the existence of two first integrals[8]

$$H = \frac{1}{2}\|\dot{q}\|^2 - \frac{m_A}{\|q_A\|} - \frac{m_B}{\|q_B\|}, \tag{3}$$

$$G = \langle q_A \wedge \dot{q} | q_B \wedge \dot{q} \rangle - \frac{m_A}{\|q_A\|}\langle q_A | u \rangle + \frac{m_B}{\|q_B\|}\langle q_B | u \rangle. \tag{4}$$

We use $\langle . | . \rangle$ for the inner product and $\wedge$ for the exterior product. Most readers would prefer to read '$\wedge$ for the vector product', and this is possible: it also gives the correct formula. But 'exterior product' is more accurate, as it works in dimension 2, 3, etc., not only in dimension 3, and does not require a conventional orientation of the 3D space. As the exterior product does not even require a choice of Euclidean structure, it will be useful for affine dynamics. Remember simply that $q_A \wedge \dot{q}$ is called a bivector, it has 1 coordinate in dimension 2, 3 coordinates in dimension 3, 6 coordinates in dimension 4, etc.

The expansion of the first term of G is $\langle q_A | q_B \rangle \|\dot{q}\|^2 - \langle q_A | \dot{q} \rangle \langle q_B | \dot{q} \rangle$. But the expression $\langle q_A \wedge \dot{q} | q_B \wedge \dot{q} \rangle$ for this term is better because it makes clear that it is an indefinite quadratic form in $\dot{q}$ whose null directions pass by the points A and B; and this may suggest the introduction of elliptic coordinates in the resolution. We will not go further in the question of the resolution. We simply note that the linear combinations $G + \|u\|^2 H$ and $G + \|u\|^2 H/2$ play some role in the study of the problem.

The first idea that comes in investigating possible generalizations of (2) is to change the force law, giving two arbitrary real functions ϕ_A and ϕ_B and considering

$$\ddot{q} = -m_A \phi_A(\|q_A\|^2) q_A - m_B \phi_B(\|q_B\|^2) q_B. \tag{5}$$

As was already noticed by Lagrange, nothing replaces the first integral G in this general case. The energy H takes the form $2H = \|\dot{q}\|^2 + m_A \Phi_A(\|q_A\|^2) + m_B \Phi_B(\|q_B\|^2)$, where Φ_A and Φ_B are such that $\Phi'_A = \phi_A$ and $\Phi'_B = \phi_B$.

A new remark is that there exists a kind of dual possibility of generalization. If we forget the Euclidean structure of the plane and define the equation

$$\ddot{q} = -m_A f_A(q_A) q_A - m_B f_B(q_B) q_B \tag{6}$$

where f_A and f_B are quite arbitrary positively homogeneous functions of degree -3, then we lose the integral H, while G takes a more general form that we shall write after some explanation about the one-fixed-centre problem.

[8] In dimension 3 the angular momentum with respect to the (AB) axis is conserved. We can express it as the trivector $C = q_A \wedge q_B \wedge \dot{q}$. The first integrals H and G keep the same expression. It is obvious that H, G and C commute.

4. One centre: a problem integrated by Jacobi

We describe in a modern language a discovery by Jacobi [Jac] that has been developed later by Darboux [Da1]. The literature on the subject is very brief, apparently limited to the few pages of these authors, and to the short presentations in the treatises [App], [Rou] and [Whi]. However, all the recent works trying to 'explain' the existence of the eccentricity vector, discussing a 'hidden symmetry', should be reconsidered in view of the Jacobi–Darboux generalization. Jacobi studied the equation

$$\ddot{q} = -mf(q)q, \tag{7}$$

where f is any positively homogeneous function of degree -3, everywhere defined except at the origin. Intuitively, f decreases as $1/\|q\|^3$ when we follow the rays, but behaves differently as a function of the 'angle'. We should not consider that we can measure angles, but rather think our plane of motion as a vector space of dimension 2 where no Euclidean structure is distinguished. A vector space is an affine space where a particular point called zero has been distinguished.

A way to state the Jacobi result is: *system (7) is integrable by quadratures. It possesses two or three independent first integrals of motion, of degree 1 or 2 in the velocities.*

The constant of areas $\mathcal{C} = q \wedge \dot{q}$ is a first integral: $\dot{\mathcal{C}} = q \wedge \ddot{q} = 0$. This is the first theorem in Newton's *Principia*: the area law is true for any kind of central force. The name 'angular momentum' for $\mathcal{C}$ is quite inappropriate in a context where we did not choose any way to measure angles. It is only in particular cases that the conservation of $\mathcal{C}$ can be related to some symmetry of rotation.

4.1. Linear coordinates

From now on it is a reasonable choice, in order to stay elementary, to give up the vectorial notations. We choose a system of linear coordinates (x, y) of our 2-dimensional vector space. The coordinates of the vector q being x and y, our system and integral read

$$\ddot{x} = -mf(q)x, \qquad \ddot{y} = -mf(q)y, \qquad C = x\dot{y} - y\dot{x}. \tag{8}$$

The previous formula $\mathcal{C} = q \wedge \dot{q}$ is slightly different from the new one: $\mathcal{C}$ was a bivector; now C is a real number. This means that we implicitly chose a unit of area. To obtain the real number from the bivector, it is sufficient to evaluate the bivector against $\mathrm{d}x \wedge \mathrm{d}y$. This is the elementary computation $\langle \mathrm{d}x \wedge \mathrm{d}y | q \wedge \dot{q} \rangle = \langle \mathrm{d}x|q\rangle\langle \mathrm{d}y|\dot{q}\rangle - \langle \mathrm{d}x|\dot{q}\rangle\langle \mathrm{d}y|q\rangle = x\dot{y} - y\dot{x}$.

Making a choice of linear coordinates (x, y) we chose a unit of area. This is a serious dilemma for us. In the same way, the choice of the linear coordinates (x, y) implicitly distinguishes among formulae (1) the 'standard distance' with $p = r = 1, q = 0$. While working in coordinates, we cannot immediately decide if the formulae we write depend or not on the choice of a Euclidean structure, i.e. if they are natural constructions in affine geometry.

This is why vector notations would be better in these considerations: they display clearly what the ingredients are. But, on the other hand, vector notations may render the text unreadable[9]. We will soon take party for the linear coordinates and let the reader convince himself that no choice of Euclidean structure is implicitly introduced, what he can do using linear changes of coordinates, or computing the coordinates of the vectorial expressions in [Al1].

[9] Between coordinates and exterior algebra notations there is the possibility of working with the traditional notations with indices. An outstanding discussion of tensor notations is [Sch], section 11. Avoiding vector notations we follow in a sense a recommendation in a footnote of section 11.7: 'But following the advice of F Klein who warned him that the paper would never be read, he translated it in terms of RICCI calculus'.

4.2. Other first integrals

To express the other first integrals, we should first compute an important vector k which, up to a conventional scaling, depends only on the function f. Let $\sigma = x\,\mathrm{d}y - y\,\mathrm{d}x$ be the 1-form measuring twice the infinitesimal area swept out from the origin by the moving point q with coordinates x and y. In vector notation this form is related to the constant of areas C by the formula $C = \langle \sigma | \dot{q} \rangle$. Note that we have $\sigma = q \lrcorner\, \mathrm{d}x \wedge \mathrm{d}y$ and the elementary 'interior product' versus 'exterior product' (see [Ste], p 20) formula $C = \langle q \lrcorner \mathrm{d}x \wedge \mathrm{d}y | \dot{q} \rangle = \langle \mathrm{d}x \wedge \mathrm{d}y | q \wedge \dot{q} \rangle$.

The coordinates of the vector k are

$$x_k = \oint x f(q)\sigma, \qquad y_k = \oint y f(q)\sigma. \tag{9}$$

The integration path is any simple, direct, closed path in the plane, around the origin. The result of each integration does not vary if we vary the path without crossing the origin: the 1-forms $xf(q)\sigma$ and $yf(q)\sigma$ are closed. This follows from the standard lemma:

Lemma. *Let $g(x, y)$ be any positively homogeneous function of degree -2, and $\sigma = x\,\mathrm{d}y - y\,\mathrm{d}x$. Then $\mathrm{d}(g\sigma) = 0$, i.e. $g\sigma$ is a closed 1-form.*

Proof. We have $g\sigma = -gy\,\mathrm{d}x + gx\,\mathrm{d}y$ and we must check whether $-\partial(gy)/\partial y = \partial(gx)/\partial x$. This happens to be Euler's relation $x(\partial g/\partial x) + y(\partial g/\partial y) = -2g$, satisfied by the homogeneous function g. □

If $k \neq 0$, k points a direction that does not depend on the choice of the linear coordinates. There is a clever way to choose new linear coordinates: take $y' = y_k x - x_k y$ as the new ordinate, and $x' = ax + by$ with a and b satisfying $\lambda = ax_k + by_k \neq 0$. Then $\sigma' = x'\,\mathrm{d}y' - y'\,\mathrm{d}x' = \lambda\sigma$. In the new system, the vector $k' = \lambda k$ has the coordinates

$$x'_{k'} = \oint x' f(q)\sigma', \qquad y'_{k'} = \oint y' f(q)\sigma' = 0.$$

We select once for all this new system of coordinates and forget the $'$ symbols: the last relation will be written simply $y_k = \oint y f(q)\sigma = 0$. As the origin $q = 0$ is the unique singularity of the 1-form $yf(q)\sigma$, this relation means that the closed 1-form is exact, i.e. that there exists a function Y of q such that $\mathrm{d}Y = f(q)y(x\,\mathrm{d}y - y\,\mathrm{d}x)$. Consequently $\dot{Y} = \langle \mathrm{d}Y, \dot{q} \rangle = f(q)yC$, in which we recognize one of the second members in (8). We have $m\dot{Y} = -C\ddot{y}$. Thus we get a first integral by integration. There exists a constant β such that

$$\beta = mY(q) + C\dot{y}. \tag{10}$$

This is our second invariant quantity. Observe that Y is positively homogeneous of degree 0, i.e. constant along the rays from the origin $q = 0$.

4.3. Integration by quadratures

The particular case where $x_k = \oint x f(q)\sigma$ is also zero will be studied with attention, but even when $x_k \neq 0$ we can obtain a relation similar to (10)

$$\alpha = mX(q) + C\dot{x}. \tag{11}$$

The fundamental difference is that here the 'function' $X(q)$, constructed by integration along paths of the closed form $xf(q)\sigma$, is many-valued. If its value at q_0 is $X(q_0)$, and if we move the point, making one turn counterclockwise around the origin, the value of X at the same point q_0 will be

$$X(q_0) + \oint_{q_0}^{q_0} x f(q)\sigma = X(q_0) + x_k.$$

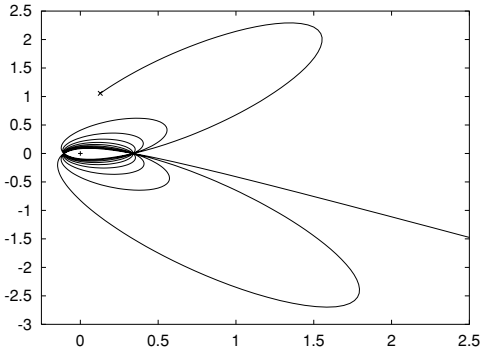

Figure 2. An orbit of a Jacobi problem.

The pseudo first integral (11) will allow us to show that any solution of system (8) can be obtained by a quadrature.

We take the initial condition $(q_0, \dot{q}_0)$. If $\mathcal{C} = q_0 \wedge \dot{q}_0 = 0$, the trajectory is rectilinear; it is a trajectory of the usual Kepler problem. If $C \neq 0$ we know that the trajectory will 'turn' in a given direction, without going backwards. It cuts the rays transversally. The set of rays 'parametrizes' the trajectory. One can of course parametrize by a number rather than by a set, e.g. choosing, as Jacobi did, the angle of the ray with a fixed direction. But in the affine context the definition of an angle implies an arbitrary choice.

To obtain the position q and the velocity $\dot{q}$ when the trajectory crosses a given ray we first compute X and Y on this ray. The quantities α, β and $C \neq 0$ being computed from the initial condition $(q_0, \dot{q}_0)$, we deduce from (10) and (11) the coordinates $(\dot{x}, \dot{y})$ of $\dot{q}$. The 'length' of q is now determined by the equation $C = x\dot{y} - y\dot{x}$. Thus we obtained $(q, \dot{q})$. We used a 'quadrature' at the first step: the determination of X and Y for the chosen ray. Another quadrature is required to obtain the time t when we cross the chosen ray. For this we compute twice the area swept out, and divide by the constant of areas C.

Let us come back to our statement of Jacobi's result. The integrability by quadratures has been established. But we have also a claim on the number of first integrals. We know C and β, respectively a linear function and a quadratic function in $\dot{q}$. In the case $k \neq 0, \alpha$ is not a first integral. There is no apparent obstruction to the existence of a third invariant quantity but it is easy to see that it could not be a quadratic function in $\dot{q}$. Instead, α is a first integral in the case $k = 0$, where X is a 'good' single-valued function.

4.4. Orbits

The Jacobi problem is thus integrable. Modern physicists would tend to reject it as 'not physical', because the position space is neither naturally Euclidean nor pseudo-Riemannian, and the phase space is not naturally symplectic. However it is objectively as strongly related to the Kepler problem as are the generalizations integrated by Newton, where the magnitude of the central force is an arbitrary function of the distance. We do think it occupies an important place in the theory of integrable systems, being historically one of the first 'physical-like' integrable systems that does not live in a phase space canonically endowed with a symplectic form. The nonholonomic mechanical systems provide more physical examples (see [BaC], [BM1], [VeV]).

We give in figure 2 a typical orbit of the Jacobi problem in the case $k \neq 0$. Conventionally the direction of k is horizontal. One of the striking features is the existence of two points

where the trajectory passes several times. An elementary discussion of the states compatible with given invariants $C = C_0 \neq 0$ and $\beta = \beta_0$ will give us the coordinates of these points.

The result of this discussion is: to each position out of the horizontal axis is associated a unique velocity vector such that $C = C_0 \neq 0$ and $\beta = \beta_0$; to a position on this axis no compatible velocity vector can be associated, with the exception of at most one position with $x > 0$, and at most one position with $x < 0$. To each of these exceptional positions are associated infinitely many velocity vectors.

The proof is: given $(x, y) \neq (0, 0)$, $C_0 \neq 0$ and β_0, the condition $x\dot{y} - y\dot{x} = C_0$ fixes a line in the $(\dot{x}, \dot{y})$ space. The condition $\beta_0 = mY(x, y) + C_0\dot{y}$ uniquely determines $\dot{y}$: except if the line is horizontal, we obtained a unique $(\dot{x}, \dot{y})$ on it. The line is horizontal when $y = 0$; in this case we have $\dot{y} = [\beta_0 - mY(x, 0)]/C_0$. But Y is constant on the rays: this equation may be written $\dot{y} = \dot{y}_+ \in \mathbb{R}$ if $x > 0$, $\dot{y} = \dot{y}_- \in \mathbb{R}$ if $x < 0$. This forces two values for x: $x_+ = C_0/\dot{y}_+$ and $x_- = C_0/\dot{y}_-$. If $x_+ > 0$, we get an exceptional position $(x_+, 0)$. If $x_- < 0$, we get an exceptional position $(x_-, 0)$.

5. The Jacobi–Darboux attractors

The case $k = 0$, i.e. $x_k = y_k = 0$ in (9), defines an important subclass of Jacobi problems, which includes the Kepler problem. The construction leading to the expression (10) of the invariant scalar β gives more. It gives an invariant vector with coordinates (α, β).

Let (x, y) be the coordinates of the position q in a system of linear coordinates, $\sigma = x\,\mathrm{d}y - y\,\mathrm{d}x$ the 'area swept' 1-form, $\ddot{x} = -mf(q)x$ and $\ddot{y} = -mf(q)y$ the equations of motion. Here f is a positively homogeneous function of degree -3 such that $x_k = y_k = 0$, i.e. such that the forms $xf(q)\sigma$ and $yf(q)\sigma$ are exact on the plane minus the origin. This data defines what we call a *Jacobi–Darboux attractor*. We introduce two functions $X(q)$ and $Y(q)$ characterized, up to the addition of an arbitrary constant, by the equations

$$\mathrm{d}X = f(q)x(x\,\mathrm{d}y - y\,\mathrm{d}x), \qquad \mathrm{d}Y = f(q)y(x\,\mathrm{d}y - y\,\mathrm{d}x).$$

As above, we deduce that the quantities

$$C = x\dot{y} - y\dot{x}, \qquad \alpha = mX(q) + C\dot{x}, \qquad \beta = mY(q) + C\dot{y}, \tag{12}$$

are first integrals of the motion. We note that $x\beta - y\alpha = C^2 + m(xY - yX)$ is an equation for the trajectory, which must be closed and without self-intersection if it makes more than one turn around the origin, and unbounded otherwise.

One of the simplest Jacobi–Darboux problems is the Kepler problem, where $f = (x^2 + y^2)^{-3/2}$. We choose $X = (x^2 + y^2)^{-1/2}y$ and $Y = -(x^2 + y^2)^{-1/2}x$. What is usually called the eccentricity vector[10] is $m^{-1}(-\beta, \alpha)$ or $m^{-1}(\beta, -\alpha)$.

6. Two Jacobi–Darboux attractors

We consider equation (6), which corresponds to the attraction of two fixed centres of Jacobi's type. We write this equation in coordinates. The attractor A is at $(1, 0)$, the attractor B at $(-1, 0)$ and q at (x, y). Let $x_A = x - 1$, $x_B = x + 1$, $q_A = (x_A, y)$, and $q_B = (x_B, y)$. System (6) reads:

$$\ddot{x} = -m_A f_A(q_A)x_A - m_B f_B(q_B)x_B, \qquad \ddot{y} = -m_A f_A(q_A)y - m_B f_B(q_B)y. \tag{13}$$

[10] Also called perivector, or Laplace–Runge–Lenz vector, but due to Jacob Herman, who wrote one of its coordinates in 1710, and to Lagrange, who published the expressions of its three coordinates in his famous paper [La1], p 132.

The function f_A and f_B are positively homogeneous of degree -3. To each centre we can associate the quantities introduced in the study of Jacobi's one centre problem. We put $\sigma_A = x_A\,\mathrm{d}y - y\,\mathrm{d}x_A$ and $\sigma_B = x_B\,\mathrm{d}y - y\,\mathrm{d}x_B$.

Theorem. *If both closed 1-forms $f_A(q_A)y\sigma_A$ and $f_B(q_B)y\sigma_B$ are exact forms, i.e. if there exist two single-valued functions $Y_A(q_A)$ and $Y_B(q_B)$ such that*

$$\mathrm{d}Y_A = f_A(q_A)y\sigma_A, \qquad \mathrm{d}Y_B = f_B(q_B)y\sigma_B,$$

System (13) admits a first integral G. Let $C_A = x_A\dot{y} - y\dot{x}_A$, $C_B = x_B\dot{y} - y\dot{x}_B$. Then

$$G = C_AC_B + 2m_AY_A(q_A) - 2m_BY_B(q_B). \tag{14}$$

Proof. This is easily checked using the following straightforward identities: $\dot{C}_A = 2m_Bf_By$, $\dot{C}_B = -2m_Af_Ay$, $\dot{Y}_A = f_AyC_A$, $\dot{Y}_B = f_ByC_B$. □

The hypothesis of the theorem may be stated differently: the vectors k_A and k_B associated with both centres by expressions corresponding to (9) are horizontal. This is true if both centres are Jacobi–Darboux centres, and we dedicated our numerical studies to this particular case. If equation (13) is equation (2) the expressions (4) and (14) for G coincide: we have $f_A = \left(x_A^2 + y^2\right)^{-3/2}$, $f_B = \left(x_B^2 + y^2\right)^{-3/2}$, and we choose $Y_A = -\left(x_A^2 + y^2\right)^{-1/2}x_A$, $Y_B = -\left(x_B^2 + y^2\right)^{-1/2}x_B$.

7. Numerical study of some examples: near-integrable behaviour

We report our numerical exploration of these generalized Euler's problems, showing three examples that seem to us significant. In all cases we met, the result is either escape or near-integrable behaviour. The third experiment displays a set of islands whose complexity seems incompatible with integrability.

We present Poincaré sections and orbits in the plane. In each example we fix once for all the value of the first integral G. The Poincaré section is $y = 0$, with the condition $\dot{y} > 0$. In the third example, we chose to display only the points with $x < 0$. Since the examples have very large orbits, we have taken throughout the numerical experiments a somewhat arbitrary cut-off criterion given by the value of any coordinate or velocity greater than a thousand.

We work with system (13) that we write in the simplified form

$$\ddot{x} = -f_A(x_A, y)x_A - f_B(x_B, y)x_B, \qquad \ddot{y} = -f_A(x_A, y)y - f_B(x_B, y)y, \tag{15}$$

with $x_A = x - 1$, $x_B = x + 1$. In the three examples the functions f_B are different, but the function f_A is the same. Here is its expression, followed by the expression of its 'primitive' Y_A, which appears in the expression of G.

$$f_A(\xi, \eta) = \frac{3\xi\eta}{10r^5} + \frac{1}{2r^3}, \qquad Y_A(\xi, \eta) = \frac{\eta^3}{10r^3} - \frac{\xi}{2r}, \qquad \text{with } r = \sqrt{\xi^2 + \eta^2}.$$

7.1. First example

In figures 3 and 4 the function f_B and its primitive are

$$f_B(\xi, \eta) = \frac{2\xi(\xi^2 - 3\eta^2)}{r^6} + \frac{1}{r^3}, \qquad Y_B(\xi, \eta) = \frac{\xi^4 - 3\eta^4 + 6\xi^2\eta^2}{4r^4} - \frac{\xi}{r}.$$

We chose $G = 5.504331$ for all the orbits. We observe that the Poincaré map is close to a linear map, on a whole domain delimited by the escape criterion.

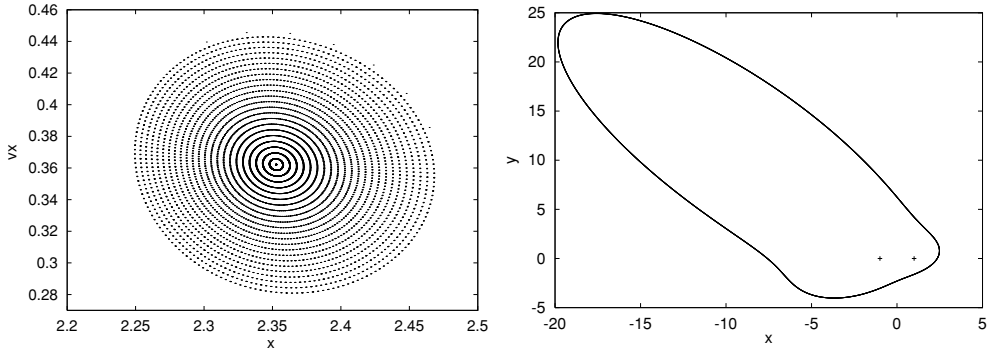

Figure 3. Example 1. Left side: Poincaré section ($y = 0, \dot{y} > 0$), from the central periodic orbit to the neighbourhood of the last torus before cut-off. Right side: the periodic orbit $x = 2.352\,464$, $\dot{x} = 0.362\,169$ and $\dot{y} = 1.050\,5775$.

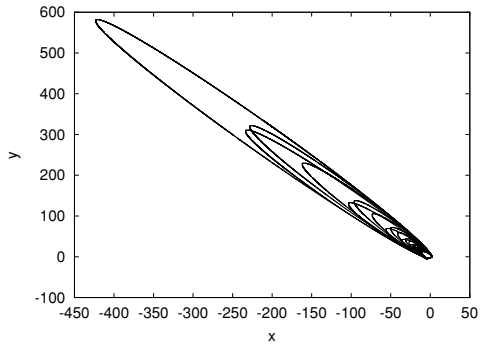

Figure 4. Example 1. The last orbit shown in the Poincaré section just before our escape criterion is satisfied: $x = 2.469\,875$, $\dot{x} = 0.362\,169$ and $\dot{y} = 0.990\,5488$.

7.2. Second example

In figures 5 and 6, we have chosen

$$f_B(\xi, \eta) = \frac{12\xi^2\eta^2}{(\xi^4 + \eta^4)^{7/4}}, \qquad Y_B(\xi, \eta) = \frac{-4\xi^3}{(\xi^4 + \eta^4)^{3/4}}.$$

We have $G = 10.92$ for all the orbits. Here the section displays a wide domain with strong torsion but does not show any indication of non-integrability.

7.3. Third example

In figures 7–9, we have fixed $G = 1.55230\,255$ and chosen

$$f_B(\xi, \eta) = \frac{\xi\eta}{r^5} + \frac{1}{r^3}, \qquad Y_B(\xi, \eta) = \frac{\eta^3}{3r^3} - \frac{\xi}{r}.$$

The system behaves as a typical conservative system close to an integrable one.

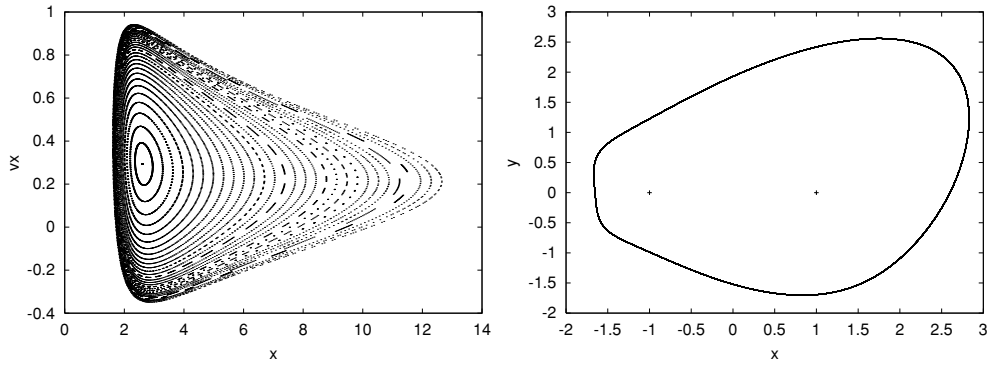

Figure 5. Example 2. Left side: Poincaré section ($y = 0$, $\dot{y} > 0$), from the central periodic orbit to the neighbourhood of the last torus before cut-off. Right side: the periodic orbit with initial conditions $x = 2.6004$, $\dot{x} = 0.2935$ and $\dot{y} = 0.824\,809$.

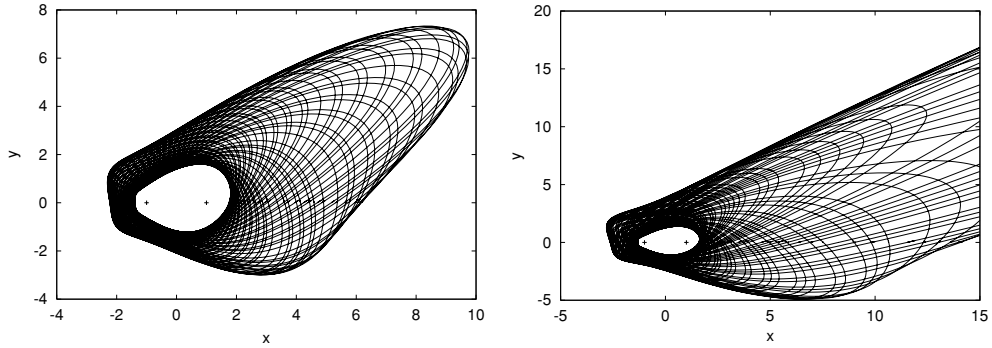

Figure 6. Example 2. Left side: a typical torus at $x = 6.1$, $\dot{x} = 0.2935$ and $\dot{y} = 0.329\,025$. Right side: a detail of the last torus before escape with $x = 13.6$, $\dot{x} = 0.2934$ and $\dot{y} = 0.145\,976$.

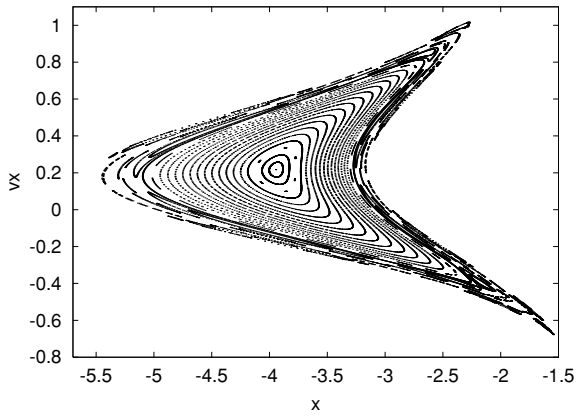

Figure 7. Example 3. Section ($x < 0$, $y = 0$, $\dot{y} > 0$) starting at the central periodic orbit up to escaping orbits.

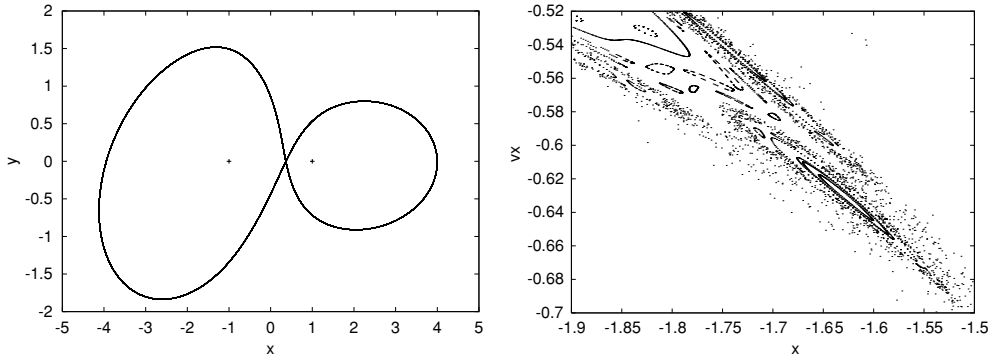

Figure 8. Example 3. Left: the central periodic orbit $x = -3.946\,1335$, $\dot{x} = 0.217\,055\,17$ and $\dot{y} = 0.418\,511\,07$. Right: detail of the section above, with islands.

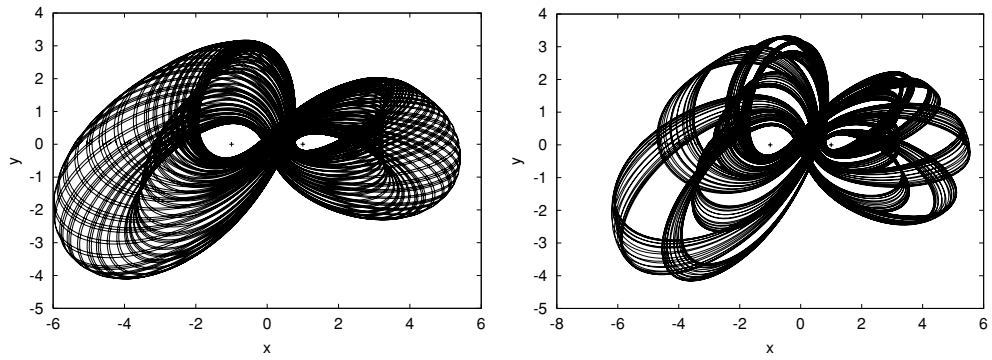

Figure 9. Example 3. Left: a regular torus, with initial coordinates $x = -1.808\,57$, $\dot{x} = -0.526\,993\,42$, $\dot{y} = 1.060\,1435$. Right: an 'island', corresponding to initial conditions $x = -1.564\,1944$, $\dot{x} = -0.663\,632\,63$, $\dot{y} = 1.328\,239$.

8. Elements of interpretation

8.1. Behaviour at infinity

Example 1 displays a spectacular deformation of the orbit, which extends to infinity, while the behaviour in the Poincaré section remains perfectly quiet. A kind of explanation for this phenomenon comes from geometrical considerations. We have chosen the plane as the domain for the motion, but there is a natural bigger domain for this kind of systems. In mathematical terminology, it is the double covering of the projective closure of the plane. More simply, it is the space whose points are half-lines drawn from the origin in a three-dimensional vector space V. It has the topology of the sphere. Our plane is one half of this 'sphere': we consider it as an affine plane in V, and we identify any of its points to the half-line drawn through it from the origin of V. In this way we only get a 'half' of the space of half-lines, a hemisphere in our sphere. Escaping orbits appear as orbits cut by the boundary of the hemisphere (in the classical Kepler problem, hyperbolas appear in the same way as cut ellipses). These remarks lead to 'projective dynamics'. Some elements may be found in [Ap1], [Alb] and [Al1]. The conclusion is the following: in figures 3 and 4 the last orbits, corresponding to the exterior

circle in the Poincaré section, are special in affine dynamics, as they are the first to escape, but are ordinary orbits in the 'good' framework, that would be projective dynamics.

8.2. Regular behaviour

In the examples most trajectories are very regular. To explain this we must describe the systems, in the domains we studied them, as small perturbations of integrable systems.

This raises a first question: what are the integrable systems nearby? We know very few cases where our generalized Euler problem is integrable, namely the classical case and its projective transformations. These projective transformations are not well known, and also belong to the 'projective' aspects of affine dynamics, discovered by Appell. Here is their description. From a dynamics in the space of half-lines of V, mentioned above, we obtain an affine dynamics on any affine plane included in V. If for such a plane the dynamics is defined by Euler's equations (2), then on another plane it will be defined by different equations, namely equations of the family (13) with $f_A(\xi, \eta) = (p\xi^2 + 2q\xi\eta + r\eta^2)^{-3/2}$, $f_B(\xi, \eta) = (p'\xi^2 + 2q'\xi\eta + r'\eta^2)^{-3/2}$, where $p, \ldots, r'$ are 6 real numbers. These are the projective transformations. They are integrable.

This family seems too small to provide good approximations of our systems. Maybe a supplementary explanation for the regularity of most trajectories is our implicit requirement that the trajectory should not escape. Maybe the boundedness on a long time is satisfied only for orbits sufficiently close to orbits of an integrable system.

8.3. About Hamiltonianity

Our third Poincaré section looks like the usual sections of near-integrable Hamiltonian systems. One can distinguish on the pictures some familiar objects: many periodic orbits of different complexity, invariant closed curves, probably also stable and unstable manifolds near hyperbolic periodic points. However we do not think that our systems are Hamiltonian, even if we cannot provide any convincing argument (compare [BoM]).

Let us call a natural system a differential system $\ddot{x} = X(x, y)$, $\ddot{y} = Y(x, y)$, with the property that the field of forces (X, Y) is derived from a potential U, i.e. $X = \partial U/\partial x$ and $Y = \partial U/\partial y$. These systems are well studied; they are Hamiltonian systems; in particular they possess a first integral $\dot{x}^2 + \dot{y}^2 - 2U$.

In system (13) the existence of a potential is not assumed. However, there is a first integral G, which is 'quadratic', i.e. is a polynomial of degree two in the velocity vector $(\dot{x}, \dot{y})$. This property makes our system a close generalization of natural systems. This kind of generalization has not been well studied, even if by many aspects it is much closer to natural systems than other well-known generalizations. It presents striking formal similarities with natural systems. For example, by an argument due to Bertrand [Ber], the first integral $G = C_A C_B + 2W(x, y)$ uniquely determines the field of forces, by the relations

$$-y(xY - yX) + \frac{\partial W}{\partial x} = 0, \qquad (x^2 - 1)Y - xyX + \frac{\partial W}{\partial y} = 0.$$

We would like to go further and find a symplectic form such that our system is the Hamiltonian system associated with G. This seems impossible, but it is however possible to obtain something if we accept to make a change of time and if we avoid the $y = 0$ axis. We will not develop here these interesting properties, because we can explain the main features of our Poincaré sections by much simpler considerations.

An area is preserved on the Poincaré section. Indeed our system defines a first-order differential system in the phase space $\mathbb{R}^4$ of the form $\dot{x} = v_x$, $\dot{y} = v_y$, $\dot{v}_x = X(x, y)$, $\dot{v}_y = Y(x, y)$. Such

a system is obviously divergence free. This property, together with the invariance of G, implies that the first return map on the Poincaré section is area-preserving. Precisely, the area form $(x^2-1)^{-1}\,\mathrm{d}x \wedge \mathrm{d}v_x$, defined on the Poincaré section $y = 0, x^2 - 1 > 0$ and $G = G_0$, is invariant. The well-known theory of area-preserving maps provides a good framework to prove, for example, the existence of invariant closed curves, using KAM theory.

Our systems are reversible. Any system of the form $\ddot{x} = X(x, y)$, $\ddot{y} = Y(x, y)$ is reversible: if at some instant we change the velocity $(\dot{x}, \dot{y})$ in $(-\dot{x}, -\dot{y})$, the particle will go backwards on its trajectory. This property is easier to observe than the existence of a preserved area, and, according to [Mos], the framework of reversible systems will also simplify the proofs in KAM theory. KAM theory has been developed initially in the framework of Hamiltonian systems. But even today it is not well-known that Bibikov and Pliss, and Moser independently, showed in 1967 that this framework might be replaced, and that one could work with reversible perturbations of integrable systems as well (see [Sev]). The theory remains very similar, and also predicts the existence of invariant closed curves in our Poincaré sections.

Acknowledgments

We thank H Cabral and the Departamento de Matemática da Universidade Federal de Pernambuco for their hospitality and help, C Simó for his rk78 code and his very useful comments, C Azevedo, I Metzmeyer, C Alunni, J Briggs, A Borisov, A Chenciner, A Fathi, J Howard, J Laskar, A Neishtadt and M Sevryuk for precious information.

References

[Alb] Albouy A 2002 Lectures on the two-body problem *Classical and Celestial Mechanics. The Recife Lectures* ed H Cabral and F Diacu (Princeton, NJ: Princeton University Press) pp 63–116

[Al1] Albouy A 2003 The underlying geometry of the fixed centers problems *Topological Methods, Variational Methods and their Applications* ed H Brezis, K C Chang, S J Li and P Rabinowitz (Singapore: World Scientific) pp 11–21

[Ap1] Appell P 1891 Sur les lois de forces centrales faisant décrire à leur point d'application une conique quelles que soient les conditions initiales *Am. J. Math.* **13** 153–8

[App] Appell P 1909 *Traité de Mécanique Rationnelle* 3rd edn vol 1 (Paris: Gauthier-Villars) section 224

[BaC] Bates L and Cushman R 1999 What is a completely integrable nonholonomic dynamical system? *Rep. Math. Phys.* **44** 29–35

[Bel] Béletski V 1977/1986 *Essai sur le mouvement des corps cosmiques* (Moscou: Mir) deuxième essai

[Ber] Bertrand J 1857 Mémoire sur quelques-unes des formes les plus simples que puissent présenter les intégrales des équations différentielles du mouvement d'un point matériel *J. Math. Pures Appl.* **2** 113–40

[BoM] Borisov A V and Mamaev I S 2002 Obstacle to the reduction of nonholonomic systems to the Hamiltonian form *Dokl. Phys.* **47** 892–4

[BM1] Borisov A V and Mamaev I S 2002 Rolling of a rigid body on plane and sphere Hierarchy of dynamics *Regul. Chaot. Dyn.* **7** 177–200

[Dar] Darboux G 1901 Sur un problème de mécanique *Arch. Néerlandaises Sci.* **6** 371–6 (also presented in [Whi], section 152)

[Da1] Darboux G 1884 Sur une loi particulière de la force signalée par Jacobi *Note 11 au cours de mécanique de Despeyrous* tome 1er (Paris: Hermann)

[Dem] Demin V G 1961 Orbits in the problem of two fixed centers *Sov. Astron.* **4** 1005–12

[Eul] Euler L 1748/1988 *Introduction to Analysis of the Infinite* (Berlin: Springer) translated by J D Blanton, book 2, chapter 18

[Eu1] Euler L 1764/1765/1760 De motu corporis ad duo centra virium fixa attracti, Problème. Un corps étant attiré en raison réciproque quarrée des distances vers deux points fixes donnés trouver les cas où la courbe décrite par ce corps sera algébrique. *Opera Omnia* **2–6** 209–46, 247–73, 274–93

[Eu2] Euler L 1762/1768 Correspondance de Lagrange avec Euler œ*uvres de Lagrange* vol 14, Lettre du 9 novembre, p 204 Lettre du 16 février, p 213

[HoW] Howard J E and Wilkerson T D 1995 Problem of two fixed centers and a finite dipole: A unified treatment *Phys. Rev.* A **52** 4471–92

[Jac] Jacobi C G J 1842/1969 *De motu puncti singularis* Werke Bd 4 (New York: Chelsea) pp 263–88 (p 282)

[Kil] Killing W 1885 Die Mechanik in den Nicht-Euklidischen Raumformen *J. für die reine Angew. Math.* **98** 1–48

[KoH] Kozlov V V and Harin A O 1992 Kepler's problem in constant curvature spaces *Celest. Mech. Dyn. Astron.* **54** 393–9

[Lag] Lagrange J L 1766–69/1892/1973 *Recherches sur le mouvement d'un corps qui est attiré vers deux centres fixes* œ*uvres* vol 2 (Paris: Gauthier-Villars/Hildesheim: Olms) pp 67–121

Lagrange J L 1889 *Mécanique Analytique,* œ*uvres,* 4th edn, vol 12 (Paris: Gauthier-Villars) p 101

[La1] Lagrange J L 1781 Théorie des variations séculaires des éléments des planètes œ*uvres* **5** 125–207

[Leg] Legendre A M 1825 *Traité des Fonctions Elliptiques et des Intégrales Eulériennes, Avec des Tables Pour en Faciliter le Calcul Numérique* vol 1 (Paris: Huzard-Courcier) pp 411–539

[Lio] Liouville J 1846 Sur quelques cas particuliers où les équations du mouvement d'un point matériel peuvent s'intégrer *J. Math. pures Appl.* **11** 345–78

[Moe] Möbius A F 1827/1976 *Der barycentrische Calcul* (Leipzig: Barth/Hildesheim: Olms)

Möbius A F 1885/1967 *Gesammelte Werke* vol 1 (Leipzig: Hirzel/Wiesbaden: Sändig)

[Mos] Moser J 1973 *Stable and Random Motions in Dynamical Systems, With Special Emphasis on Celestial Mechanics* (Princeton, NJ: Princeton University Press) p 48

[RoB] Rost J M and Briggs J S 1991 Saddle structure of the three-body Coulomb problem; symmetries of doubly-excited states and propensity rules for transitions *J. Phys. B: At. Mol. Opt. Phys.* **24** 4293–322

[Rou] Routh E J 1898/1960 *A Treatise on Dynamics of a Particle* (Cambridge: Cambridge University Press/New York: Dover) p 164

[Sch] Schouten J A 1954 *Ricci-Calculus, An Introduction to Tensor Analysis and its Geometrical Applications* 2nd edn (Berlin: Springer)

[Sev] Sevryuk M B 1998 The finite-dimensional reversible KAM theory *Physica* D **112** 132–47

[Ste] Sternberg S 1983 *Lectures on Differential Geometry* (New York: Chelsea)

[VeV] Veselov A P and Veselova L E 1988 Integrable nonholonomic systems on Lie groups *Math. Notes* **44** 810–19

[Wey] Weyl H 1921/1952 *Space-Time-Matter* (New York: Dover) translated by H L Brose p xiii, 16

[Whi] Whittaker E T 1937 *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, with an Introduction to the Problem of Three Bodies* 4th edn (Cambridge: Cambridge University Press) exercise 17, chapter 4